TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
HOMEWORK 5

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Homework 5.1 (Examples of infinite products of functions*)**.** Examine for which $z \in \mathbf{C}$ the following products converge absolutely and determine the largest open set $U \subset \mathbf{C}$ on which they converge locally normally.

a. $\displaystyle\prod_{n=1}^{\infty}(1+z^n)$.

b. $\displaystyle\prod_{n=1}^{\infty}\left[1+\frac{z^n}{n!}\right]$.

c. $\displaystyle\prod_{n=1}^{\infty}\cos(2^{-n}z)$.

Homework 5.2 (Further practice on infinite products)**.** Show the product

$$H(z) = z\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

converges locally normally on $\mathbf{C}$[1].

Homework 5.3 (Sine product formula)**.** The goal of this exercise is to derive the following product formula for every $z \in \mathbf{C}$:

$$\sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left[1-\frac{z^2}{n^2}\right].$$

This will be done in several steps.

a. Use the partial fraction decomposition of $\pi^2/\sin^2(\pi z)$ from Homework 3.2 to show the following identity for every $z \in \mathbf{C}\setminus\mathbf{Z}$, with a suitable notion of convergence for the series on the right hand side[2]:

$$\pi\cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

b. For $n \in \mathbf{N}$, define the function $f_n\colon \mathbf{C} \to \mathbf{C}$ by $f_n(z) := 1 - z^2/n^2$. Compute the logarithmic derivative f'/f for the assignment $f(z) := \pi z\prod_{n=1}^{\infty} f_n(z)$.

c. Conclude by comparing suitable terms.

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[1]One can show that the limit $\gamma := \lim_{n\to\infty}\sum_{k=1}^{n} 1/k - \log(n)$ exists (which is called *Euler–Mascheroni constant*). Then $\Gamma(z) := e^{-\gamma z}/H(z)$ yields an alternative representation of the Eulerian Γ-function for every $z \in \mathbf{C}$ with $\Re z > 0$.

[2]**Hint.** You may use the following fact without proof. If $D \subset \mathbf{C}$ is a domain, $(f_n)_{n\in\mathbf{N}}$ is a sequence of continuously differentiable functions $f_n : D \to \mathbf{C}$, $f : D \to \mathbf{C}$ is continuously differentiable, $f_n(z_0) \to f(z_0)$ as $n \to \infty$ for some $z_0 \in D$, and $f_n' \to f'$ locally uniformly as $n \to \infty$, then $f_n \to f$ locally uniformly as $n \to \infty$.

Homework 5.4 (Consequences of the sine product formula)**.** Use the product formula from the previous exercise to show the following statements.

a. $\dfrac{\pi}{2} = \displaystyle\prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}$.

b. $\cos(\pi z) = \displaystyle\prod_{n=1}^{\infty} \left[1 - \frac{4z^2}{(2n-1)^2}\right]$.

c. $\dfrac{\pi^2}{6} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$[3].

[3] **Hint.** Use a Taylor expansion.